Reliability controlled by the relative locations of random variables on a finite interval

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Abstract

General equations are proposed and their advantages are demonstrated regarding reliability which depends only on the relative configurations of the controlling random variables in their domain. A number of intractable problems from structural/mechanical reliability were solved easily using the derived equations and new reliability and risk assessment methodology and powerful tools have been created in cases of random variables following a homogeneous Poisson process on a finite interval. The equations give the probability of any specified gap pattern between adjacent, Poisson-distributed failures or defects, in both cases where their number is given or is unknown on a finite interval. Using the equations derived it is demonstrated that even for a moderate number of uniformly distributed random loads, the probability of existence of clusters of two or more loads at a critically small distance is substantial and should not be neglected in reliability calculations. In case of Poisson-distributed defects existing in thin fibres or wires which cannot tolerate clusters of defects during loading, the equations give the reliability of the fibre (wire) of finite length during loading. A new reliability methodology and tools have been created to transform customers' requirements into a system reliability goal: determining the minimum hazard rate that guarantees a required probability for a specified set of maintenance-free intervals. The reliability tools proposed also permit extracting useful information from data sets containing failures following a homogeneous Poisson process, in cases where the failure times are unknown. Furthermore, the equations can also be used as reliability demonstration tools in the reliability analysis: estimating the probability for a specified set of maintenance-free intervals given the hazard rate of the system or component.

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1. Introduction

There exist a number of structural/mechanical reliability problems, whose solution requires developing new, specific tools. Such is for example the problem of finding the probability of clustering of Poisson-distributed defects over a finite length \( a \) of a fibre or wire (Fig.1a). Clustering of two defects over a small critical distance \( s \) decreases dangerously the load-bearing cross section and increases the stress concentration which further decreases the load-carrying capacity. As a result, a defect configuration where two or more defects are closer than a critical distance \( s \) cannot be tolerated during loading. In this case calculating the reliability of thin one-dimensional components containing defects is reduced to calculating the probability of existence of a 'dangerous' (critical) spatial configuration of defects. The limit state condition \([1-2]\) is 'at least two defects being at a distance smaller than a specified critical distance and the random variables controlling reliability are the coordinates of the defects. This problem was solved in earlier work \([3]\), given the number of defects in the component. Here a solution is presented for the probability of clustering, in cases where the number of defects is unknown on a piece of wire of finite length.

A similar problem is related to reliability of a structural element loaded by a specified number of identical uniformly distributed random loads (Fig.1b). In this case, the failure mode is overloading, i.e., the component fails when, for example, two or more loads are applied within a relatively small interval \( s \). The probability of failure is then equal to the probability of such a load configuration which overloads a small interval of specified length (Fig.1b). A modified problem with practical importance is presented in Fig.1c. In this case failure occurs when all of the uniformly distributed radial random loads acting on the circumference of a circular structural component are applied within a specified small angle \( s \). In this case, the failure mode is, for example, excessive deformation of the structural component due to such a critical configuration (limit state) of the loads (Fig.1d).

Problems of this nature cannot be solved using conventional methods \([1,2,4,5]\) and require specific reliability tools. Accordingly, the main purpose of the present study is to propose and demonstrate the advantages of derived generic equations that allow calculating reliability dependent on the relative locations of the random variables in their common domain.

2. General equations related to reliability controlled by the relative locations of uniformly distributed random variables. Applications related to structural reliability problems.

Only configurations of uniformly distributed random variables are considered. The location of each controlling random variable can be represented as a point in the common domain and each point in the domain corresponds to a possible location of a random variable. The random variables are not necessarily identical, only their distribution in the common domain is uniform. Suppose that a domain with measure \( V \) is defined, where \( n \) uniformly distributed random variables form a safe (failure)
configuration with probability $p$. If $p_i^*$ denote the probability of safe (failure) configurations when the i-th random variable is located on the boundary of the common domain, it can be shown (the derivation will be published elsewhere) that the link between the probability $p$ and the probabilities $p_i^*$ is given by

$$pv^n = C + \int v^{n-1} \sum_{i=1}^{n} p_i^*(v) dv$$

where $C$ is an integration constant determined from the boundary conditions of the system.

Equation (1) is general and valid for any relative configuration of the random variables in their domain. In case where all of the random variables are identical, all probabilities $p_i^*$ are equal ($p_1^* = p_2^* = ... = p^*$) and equation (1) transforms into:

$$pv^n = C + \int v^{n-1} n p^*(v) dv$$

In case of distinct random variables, for example random variables of the same type but characterised by different magnitudes, the probabilities $p_i^*$ are different in general and determining the integration constant $C$ is more complicated. Thus, in a time domain $a$, depending on which random variable appears first, different boundary conditions may be present.

The application of equation (1) to real structural reliability problems can be illustrated by a simple problem for determining the reliability of the structural element with length $a$ (Fig.1b) which fails only if all of the $n$ uniformly distributed identical loads are concentrated within a small distance $s$. In this case, the critical configuration of the controlling random variables is defined as all random variables clustering within a distance $s$ provided that each random variable is uniformly distributed over the length $a$.

Since $p_1^* = p_2^* = ... = p_n^* = p^*$, and $p^* = (s/a)^{n-1}$ if one of the loads is fixed at the beginning of the segment (point 0 in Fig.1b), according to equation (2), the probability of failure can be found from

$$pa^n = C + \int a^{n-1} n(s/a)^{n-1} da = C + ns^{n-1}a$$

Since for $a = s$, $p = 1$, substituting in equation (3) yields:

$$s^n = C + ns^n$$

from which $C = -(n-1)s^n$ and the probability of failure becomes

$$p = n(s/a)^{n-1} - (n-1)(s/a)^n$$

In case where the probabilities $p_i^*$ in equation (1) cannot be calculated easily, the problem can be solved by reducing its complexity and solving series of problems involving a smaller number of random variables. This method was used to find the reliability of a structural element loaded by $n$ identical loads, which does not fail only if the distances between the adjacent loads are greater or equal to the specified distances $s_1, s_{12}, s_{23}, ..., s_{n-1n}$; $s_1 \geq 0$; $s_{ii+1} \geq 0$; $(s_1 + \sum_{i=1}^{n-1}s_{ii+1} \leq a)$, where
$s_{i+1}$ is the specified distance between adjacent loads with indices $i$ and $i+1$ and $s_1$ is the specified distance of the first load from the start of the interval. The following equation giving the reliability $p$ of the system was derived from equation (1):

$$p = \Pr(S_1 \geq s_1 \cap S_{12} \geq s_{12} \cap S_{23} \geq s_{23} \cap \ldots \cap S_{n-1n} \geq s_{n-1n}) =$$

$$= \left[1 - (s_1 + s_{12} + s_{23} + \ldots + s_{n-1n})/a\right]^n \quad (5)$$

(An alternative derivation of equation (5), where $s_1 = 0$ has been presented in earlier work [3]).

Figure 2 depicts a case of nine failures following a homogeneous Poisson distribution over a finite time interval with length $a$. According to equation (5), the probability that the four intervals $S_{ij}$ between selected adjacent pairs of failures will be greater than the specified four minimum distances $s_{ij}$ is

$$p = \left[1 - (s_{12} + s_{23} + s_{56} + s_{67})/a\right]^n .$$

If all distances are equal ($s_{12} = s_{23} = \ldots = s_{n-1n} = s$) and $s_1 = 0$ the probability that all loads will be at a minimal distance $s$ from each other becomes

$$p = \left(1 - \frac{(n-1)s}{a}\right)^n \quad (6)$$

Correspondingly, the probability that two or more loads will be located within a critical distance $s$ (the probability of overloading a critical distance (e.g. the probability of failure) is

$$p_f = 1 - \left[1 - \left(1 - \frac{(n-1)s}{a}\right)^n\right] \quad (7)$$

According to equation (5), the probability that all random loads will be at minimal distances $s$ ($s_{12} = s_{23} = \ldots = s_{n-1n} = s$; $s_1 = s$) from one another and that the first variable will be at a minimal distance $s$ from the origin is

$$p = \left(1 - \frac{ns}{a}\right)^n \quad (8)$$

The solution of the practical structural reliability problem from Fig.1c was obtained using equations (4) and (8):

$$p = n\left(\frac{s}{2\pi}\right)^{n-1} \quad (9)$$

3. Equation (5) used as a reliability tool for determining the distribution of the gaps between a given number random variables following a homogeneous Poisson process on a finite interval

The Poisson process [6-11] is often used as a statistical model for random failures. For events following a homogeneous Poisson process with intensity $\lambda$, the distribution of the distances (intervals) $t$ between the events is the exponential distribution $F(t) = 1 - e^{-\lambda t}$. The homogeneous Poisson process and the uniform distribution are closely related. Thus, a homogeneous Poisson process on a
finite interval \( a \) can be realised by first generating the number of random points \( n \) in the interval according to a Poisson distribution with parameter \( \lambda \) and then generating the \( n \) points according to the uniform distribution \([6]\). It has also been established \([6]\) that if a number \( n \) of realisations following a homogeneous Poisson process have occurred in a fixed interval, the realisations have the distribution of the order statistics \([8]\) of a sample of size \( n \) from the uniform distribution on that interval.

For a given number of random variables on a finite interval, according to equation (5), the probability that at least one of the specified distances between the adjacent variables or between the start of the interval and the first variable will be smaller than its corresponding minimum specified distance is:

\[
p = \Pr(S_{1} \leq s \cup S_{12} \leq s_{12} \cup S_{23} \leq s_{23} \cup \ldots \cup S_{n-1n} \leq s_{n-1n}) = 1 - [1 - (s_{1} + s_{12} + s_{23} + \ldots + s_{n-1n})/a]^n, \text{where } s_{1} \geq 0, s_{12} \geq 0, s_{23} \geq 0, \ldots, s_{n-1n} \geq 0.
\]

From this equation by setting: \( s_{1} = s \), \( s_{12} = s_{23} = s_{34} = \ldots = s_{n-1n} = 0 \), the cumulative distribution of the gap from the beginning of the interval to the first random variable (or between any two adjacent random variables) is obtained:

\[
\Pr(S \leq s) = F(s) = 1 - (1 - s/a)^n
\]  

(10)

It can be verified easily that for large \( n \) or small ratios \( s/a \) equation (10) transforms into the classical exponential equation

\[
\Pr(S_{ij} \leq s) = 1 - e^{-\lambda s} = 1 - e^{-(n/a)s}
\]  

(11)

The limitations of exponential equation (11) to describe correctly the distribution of the distances between a known number of Poisson-distributed random variables (e.g. failures) have already been discussed in Ref.[3]. This is also illustrated by Fig.3 where the graphs of equations (10) and (11) have been plotted (\( a = 1 \), \( s/a = 0.65 \)).

Suppose random variables (e.g. failures or defects) follow a homogeneous Poisson process, where \( \lambda \) (the parameter of the Poisson process) is the mean number of failures on the interval with length \( a \). For unknown number of Poisson-distributed random variables (defects) on a finite interval of length \( a \), the probability \( p \) that the random failures will be apart, at specified distances \( s_{ij} \) from one another has been derived on the basis of equation (5):

\[
p = \exp(-\lambda a) \left( \exp[\lambda(a - s_{1} - \ldots - s_{m-1}s_{m})] + \lambda[(a - s_{1}) - (a - s_{1} - \ldots - s_{m-1}s_{m})] + \right.
\]

\[
\left. + \lambda^{2}[(a - s_{1} - s_{12})^{2} - (a - s_{1} - \ldots - s_{m-1}s_{m})^{2}] / 2! + \right.
\]

\[
\left. + \lambda^{3}[(a - s_{1} - s_{12} - s_{23})^{3} - (a - s_{1} - \ldots - s_{m-1}s_{m})^{3}] / 3! + \ldots \right.
\]

\[
+ \lambda^{m-1}[(a - s_{1} - \ldots - s_{m-2}s_{m-1})^{m-1} - (a - s_{1} - \ldots - s_{m-2}s_{m-1} - s_{m-1}s_{m})^{m-1}] / (m-1)!
\]

(12)

For \( m = 2 \), equation (12) results in:

\[
p = \exp(-\lambda a) \left( \exp[\lambda(a - s_{1} - s_{12})] + \lambda s_{12} \right)
\]  

(13)
Equation (13) gives the probability of the following compound event: (i) no failures on the finite interval of length \( a \) or (ii) a single failure at a distance at least \( s_1 \) from the beginning of the interval or (iii) two or more failures the first of which is at a distance at least \( s_1 \) from the beginning of the finite interval and at a distance at least \( s_{12} \) from the second failure.

Equation (12) is of particular importance to the reliability analysis and to setting system level reliability requirements. It can be used to solve the important problem of setting a reliability goal on the basis of a customer-specified level of acceptable risk \( q^* \) for unscheduled maintenance and a set of minimum maintenance-free intervals \( s_{ij} \) (Fig. 4). Equation (12) is then solved numerically regarding the hazard rate \( \lambda \) where \( p \) is replaced by \( p^* = 1 - q^* \). The latter is the probability of the specified by the customer maintenance-free intervals. The solution \( \lambda^* \) is the reliability goal: the minimum hazard rate that guarantees the specified set of maintenance-free intervals \( s_{ij} \) with probability \( p^* \).

By setting \( s_1 = s_{12} = \ldots = s_{m-1m} = s \) in equation (12), the probability that the distances between \( m \) adjacent failures (defects) on a finite interval \( a \) will be at least \( s \), and also the probability that the distance to the first failure (defect) will be at least \( s \) is obtained:

\[
p = \exp(-\lambda a)\left\{\exp[\lambda(a - ms)] + \lambda[(a - s) - (a - ms)] + \lambda^2[(a - 2s)^2 - (a - ms)^2]/2! + \lambda^3[(a - 3s)^3 - (a - ms)^3]/3! + \ldots + \lambda^{m-1}[(a - (m-1)s)^{m-1} - (a - ms)^{m-1}]/(m-1)!\right\}
\]

(14)

Another important special case from equation (12) was obtained which is related to the probability that all possible Poisson-distributed defects on a piece of wire with length \( a \) will be apart, at minimum distances \( s \) from one another.

Since the maximum number of defects that can fit into a piece of wire with length \( a \) and still be apart from each other, at a distance \( s \) is \( r = [a/s] + 1 \), where \( [a/s] \) denotes the greatest integer part of the ratio \( a/s \) which does not exceed it, by setting \( s_1 = 0 \) and \( s_{12} = s_{23} = \ldots = s \) in equation (12), it can be shown that the probability that the existing Poisson-distributed defects will be apart, at distances at least \( s \) from one another is:

\[
p = \exp(-\lambda a)\left\{1 + \lambda a + \frac{\lambda^2(a - s)^2}{2!} + \ldots + \frac{\lambda^r[a - (r-1)s]^r}{r!}\right\}
\]

(15)

The probability \( q \) that two or more defects will be at a distance smaller than \( s \) is then given by \( q = 1 - p \). In case where the number of failures (defects) is not known on the finite interval, the probability that the failure-free (defect-free) interval will be smaller than a specified quantity \( s \) is given by the exponential distribution \( 1 - e^{-\lambda s} \) (this can be verified easily from equation (12)). If more than
one failure-free interval is specified however, equations (12)-(15) should be used, not the exponential equation.

4. Monte Carlo simulations

All equations have been validated by Monte Carlo simulations. For the probability that at least two loads from five identical uniformly distributed loads acting on a structural element with length \( a = 3 \text{ m} \) will be at a distance smaller than 0.05 m \( (s \leq 0.05) \), the Monte Carlo simulation yielded an empirical probability of 0.291 which was in excellent agreement with the theoretical probability \( p = 1 - (1 - 4s/a)^5 = 0.291 \) calculated from equation (7). The probability that a load-free distance of at least \( d = 0.05 \) will be present between each pair of adjacent loads is given by equation (6): \( p = 0.709 \).

For the probability that all five loads will be at a distance smaller than \( s = 0.8 \text{ m} \), equation (4) yields \( p = 5 \left( \frac{s}{a} \right)^4 - 4 \left( \frac{s}{a} \right)^5 = 0.02 \) which is in excellent agreement with the empirical probability 0.02 obtained from the Monte Carlo simulation.

For the probability that all five radial loads will be acting within a sector fraction of \( s/(2\pi) = s/a = 0.3 \) from the circumference (Fig.1d) equation (9) yields \( p = 5 \times 0.3^4 = 0.0405 \) which is in very good agreement with the empirical probability 0.0406 obtained from the Monte Carlo simulation.

The graph of equation (6) giving the probability of existence of a cluster of two or more loads at three different critically small distances \( s = 0.1 \text{ m}, s = 0.05 \text{ m} \) and \( s = 0.01 \text{ m} \) on a structural element with length \( a = 3 \text{ m} \) is given in Fig.4.

For the probability that the distances between four random events on a time interval with length \( a = 100 \text{ units} \) will be at least \( s_{12} = 12\), \( s_{23} = 20\) and \( s_{34} = 7\), the Monte Carlo simulation yielded empirical probability of 0.138 which confirmed the theoretical probability calculated from equation (5):

\[
p = \left[ 1 - \left( s_{12} + s_{23} + s_{34} \right)/a \right]^4 = 0.138.
\]

The validity of distribution (10) applied to a given number of failures following a homogeneous Poisson process on a finite interval has been verified by running the following Monte Carlo simulation. The Poisson distributed failures were generated over a very long time interval of \( L=100000 \text{ time units} \). A smaller interval of \( a = 100 \text{ units} \) has been 'monitored' for failures. The simulation was interrupted when exactly three failures were counted over the monitored finite time interval of \( a = 100 \text{ time units} \). This simulation was repeated 100000 times and the empirical probability \( \Pr(S_1 \leq s = 48) \) that the time to the first failure will be smaller than \( s = 48 \text{ time units} \) was found to be \( \Pr(S_1 \leq s = 48) \approx 0.86 \). The probability calculated from the exact equation (10) \( \Pr(S_1 \leq s = 48) \approx 0.86 \) confirmed the result from
the Monte Carlo simulation whereas the probability $\Pr(S_1 \leq s = 48) \approx 0.76$ calculated from the exponential equation (11) was significantly smaller than the true probability. The Monte Carlo simulations yielded the same value $\Pr(S_{12} \leq s = 48) = 0.86$ for the probability that the time between the first and the second failure will be smaller than $s = 48$ time units.

Equation (12) has also been verified using computer simulations regarding the probability that an interval of length $a = 100$ units will contain no failures, a single failure or two or more failures the first of which is at least $s_1 = 20$ units from the start of the interval and at least $s_{12} = 10$ apart from the second failure. The failures (defects) were assumed to be Poisson-distributed, with density $\lambda = 0.015$ failures (defects) per unit interval. For the probability that the failures (defects) on the interval with length 100 units will be at least at distances $s_1 = 20$ units and $s_{12} = 10$ units, the Monte Carlo simulation yielded $= 0.667$, which was close to the probability calculated from equation (13) which is a special case of equation (12):

$$p = \exp(-\lambda a) \left( \exp[\lambda(a - s_1)] + \lambda s_{12} \right) = 0.67$$

Equation (15) regarding the probability that an interval of length $a = 100$ units will contain defects at least $s = 45$ units apart from one another has also been verified using computer simulations. The distribution of the defects was assumed to be a Poisson distribution, with a density $\lambda = 0.015$ defects per unit length. Thus, for the probability that the defects on the interval $a = 100$ will be at least at a distance of $s = 45$ units, the Monte Carlo simulation yielded 0.62, which was close to the probability calculated from equation (15):

$$p = e^{-\lambda a} \left( 1 + \frac{\lambda^2(a - s)^2}{2!} + \frac{\lambda^3(a - 2s)^3}{3!} \right) = 0.63$$

since $r$ in equation (15) is determined from: $r = \lceil a/s \rceil + 1 = 3$.

Finally, an important problem from setting a system reliability goal was solved.

Let $s_1 = 20$ and $s_{12} = 10$ are maintenance-free periods, specified by the customer and $q^* = 0.20$ is the customer’s acceptable risk for unscheduled maintenance. In other words, the customer wants with probability $p^* = 1 - q^* = 0.80$ a maintenance-free period $s_1$ to the first failure and a maintenance-free period $s_{12}$ between the first and the second failure. Accordingly, for the length of the interval $a = 100$ and the minimum maintenance-free periods $(s_1 = 20$, $s_{12} = 10$) the minimum value of $\lambda$ was determined which guarantees with the specified probability $p^* = 0.80$ the minimum maintenance-free periods $s_1 = 20$ and $s_{12} = 10$ between failures. For a repairable system whose failures follow a homogeneous Poisson process, this means determining $\lambda^*$ (the hazard rate) that guarantees with the
specified probability \( p^* = 0.80 \) no failures, one failure or two or more failures with specified maintenance-free periods of operation \( s_1 = 20 \) and \( s_{12} = 10 \). The numerical solution of equation (13)
\[
\exp(-\lambda a) \left( \exp[\lambda (a - s_1 - s_{12})] + \lambda s_{12} \right) - p^* = 0; \ (p^* = 0.80),
\]
(which is a special case of equation (12)) yielded a reliability goal (a minimum value of the hazard rate)
\[
\lambda^* \approx 0.009
\]
which guarantees the specified probability \( p^* = 0.80 \).

5. Discussion

The excellent agreement between the Monte-Carlo simulation results and the results obtained from the solutions of equation (1) illustrates the validity and the power of the derived equation. The equation allows to calculate reliability controlled by the relative configuration of random variables in their common domain. The power of the derived equation (1) stems also from the fact that the problems it can handle are not restricted to one-dimensional problems only or to simple functions giving the distance between the locations of the random variables.

The comparison between the graphs in Fig.3 shows that the exponential distribution underestimates the probability that a Poisson-distributed random variable will be found within a specified distance (time interval) given the number of Poisson-distributed random variables on a finite interval. This underestimation, which for \( n=2 \) random variables has an absolute value of about 0.15 and for \( n=3 \) random variables about 0.1 is significant and can have a serious impact on the precision of reliability and risk assessments. In order to avoid this, in all circumstances, the exact distribution (10) should be used instead of the exponential distribution (11) when the number of random variables on the finite interval is given.

The graphs in Fig.5 show that the probability of existence of a cluster of two or more loads over the critically small distance 0.05m (the middle curve) increases rapidly with increasing the number of random loads. The analysis demonstrates that contrary to the expectations, even for a small number of random variables, the probability of existence of two or more random variable locations at a critically small distance is substantial and should always be taken into consideration during reliability calculations.

The Monte Carlo simulation and the theoretical analysis regarding the distribution of the times to failure for random failures following a homogeneous Poisson process showed that given the number of failures on the finite interval, the distribution of the time to (between) failure(s) is given by equation (10) not by the exponential distribution (11). Distribution (10) is a special case of equation (5) and is particularly useful for cases where failures following a homogeneous Poisson process have been detected (recorded) over a finite time interval \( a \) but their times are not known. For a very small \( s/a \) ratio or a large number of failures \( n \), the exponential distribution (11) gives almost identical results with the exact distribution (10). In cases of a small number of failures \( n \) and large \( s/a \) ratios however, applying the exponential distribution (11) seriously underestimates the probability of failure within a specified time.
interval, as indicated by the computer simulations and the theoretical analysis (Fig.3). This underestimation can have a serious impact on the reliability predictions.

It must be pointed out that equation (10) should only be applied in cases where the number of random variables on the interval is given (or known, or guaranteed to exist). In cases where the distribution of the random variables (e.g. failures) is a Poisson process and the number of failures is unknown on the finite interval, equation (11) describes correctly the distribution of a single gap between failures (or the time to first failure) and it should be used, not equation (10).

In cases where numerous minimum distances between failures (defects) are specified and their number is not known on the finite interval, equation (12) and its special cases equations (13), (14) and (15) should be used. Thus, the exponential distribution is a special case of equation (12) for a single specified distance (gap) to failure (or between failures). Equation (12) is more powerful than the exponential distribution because apart from the probability of existence of a single specified gap length between failures it gives the probability of any specified combination of gap lengths between adjacent failures.

Equations (5), (12), (14) and (15) form the basis of a new reliability methodology and tools for data analysis of random failures on a finite time interval. A new reliability allocation tool has been created for setting a system reliability goal on the basis of the customer's requirement. The new method determines the minimum hazard rate of a system or component that guarantees a required probability for a specified set of maintenance-free intervals. Thus, for a repairable system whose failures follow a homogeneous Poisson distribution, a special case of equation (12) was solved to determine the system reliability goal $\lambda^*$ (the minimum hazard rate) that guarantees a required probability $p^*$ of specified maintenance-free periods of operation.

The methodology developed can also be used as a reliability demonstration tool. Given the hazard rate $\lambda$, the methodology allows to estimate the probability for a specified set of maintenance-free intervals.

In case of thin fibres or wires which cannot tolerate clusters of defects during loading, equation (15) gives the reliability of the fibre (wire) during loading.

6. Conclusions

1. General equations have been derived determining reliability which depends only on the relative configurations of the controlling random variables in their domain. A number of intractable problems from structural/mechanical reliability were solved easily using the derived equations.

2. A new reliability methodology and tools have been created to transform the customer requirements into a system reliability goal: determining the minimum hazard rate that guarantees a required probability for a specified set of maintenance-free intervals.

3. Given the number of random failures on a finite interval, the probability of a set of specified minimum distances between failures is given by equation (5). The equation allows to extract useful information from
data sets containing Poisson-distributed failures, where only the number of failures but not their times are known.

4. Equation (12) and its special cases equations (13)-(15) give the probability of any specified gap pattern between adjacent, Poisson-distributed failures or defects, whose number is not known on a specified finite interval. The exponential distribution is a special case of equation (12) for a single specified time interval between failures (or to failure).

5. In case of Poisson-distributed defects along thin fibres or wires which cannot tolerate clusters of defects during loading, equation (15) gives the reliability of the fibre (wire) during loading.

6. The new reliability methodology and tools were applied to some structural reliability problems where the random variables controlling reliability are uniformly distributed. It was demonstrated that even for a moderate number of random loads, the probability of existence of clusters of two or more loads at a critically small distance is substantial and should not be neglected in reliability calculations.

7. In case of a given number of Poisson-distributed failures on a finite interval, the exact distribution (10), not the exponential distribution (11) should be used for calculating the distribution of the distances between failures. The exponential distribution (11) gives a good approximation only for small $s/a$ ratios of the specified distance and the length of the interval or for a large number of failures $n$. If the ratio $s/a$ is large and the number of failures $n$ is small, the probability calculated from the exponential equation (11) underestimates the true probability given by equation (10) which can have significant effect on the validity of reliability assessments.

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REFERENCES

Fig. 1 In cases where clustering of the locations of random variables causes failure, calculating the reliability of the components is reduced to calculating the probability of existence of a critical configuration (cluster) of locations. (a) Failure due to clustering of defects in thin wires and fibers which cannot be tolerated during loading; (b) Failure due to overloading of a critically small distance $s$ by uniformly distributed random loads; (c) Safe configuration of uniformly distributed radial loads (d) Critical configuration of the radial loads caused by clustering within a small angle $s$. The result is excessive deformation and failure of the structural component.

$$\Pr(S_{12} > s_{12} \cap S_{23} > s_{23} \cap S_{56} > s_{56} \cap S_{67} > s_{67}) =$$
$$= \left(1 - \frac{s_{12} + s_{23} + s_{56} + s_{67}}{a}\right)^9$$

Fig. 2 The probability that the distances between the specified nine adjacent failures (denoted by 'x') will be greater than the specified gap lengths is given by equation (5).
Fig. 3 The largest differences between equation (10) and the exponential equation (11) occur at large ratios \( s/\alpha \) and a small number of random variables \( n \) (\( \alpha = 1, \ s = 0.65 \)). Only the exact equation (10) agrees with the Monte Carlo simulation results.
$S_1 \ S_{12} \ S_{23}$ - Minimum maintenance-free intervals, specified by the customer

$q^*$ - Acceptable risk for unscheduled maintenance

$p^* = 1 - q^*$ - Probability of the specified maintenance-free intervals

![Diagram of time intervals with random failures and finite time interval, a]

$\Pr(\eta > \eta_2 \cap \eta_2 < \eta_3 \cap \eta_3 > \eta_4) = p^*$

$\lambda$ (hazard rate) = ?

Fig. 4. The numerical solution of equation (12) determines the hazard rate that guarantees with a customer-specified acceptable risk a set of specified minimum maintenance-free intervals.
Fig. 5 The graphs show that the probability of clustering of two or more random loads within the critically small distances 0.1 m, 0.05 m, and 0.01 m increases rapidly with increasing the number of loads. Even for a small number of loads ($n = 3$) applied on a structural component with length 3 m, the probability that at least two loads will be closer than 0.05 m is substantial.